ORIGINAL ARTICLE

RESPONSE OF BEAM TO VARIABLE ELASTIC SUBGRADE SUBJECTED TO A DISTRIBUTED HARMONIC LOADING CONDITION

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ABSTRACT

Background: The dynamic analysis of beams resting on elastic foundation is one of the important topics in engineering and applied Mathematics and is a subject of investigation for many decades. **Objectives:** The specific aim of this paper is to find the analytical solutions to the governing fourth order partial differential equation that involves the variable elastic subgrade and variable magnitude distributed load on the system. **Methods:** Use is made of the elegant method of Galerkin and convolution theory. **Results:** From the analytical and numerical solutions, as the variable elastic foundation, damping, and axial force increases, the response amplitudes of thin beam under the action of harmonic varying distributed loads with constant velocity decreases and are displayed in the form of plots. **Conclusions:** The analytical solutions to the governing fourth order partial differential equation for thin beam is solved and the dynamic effects of some vital parameters are discussed.

Keywords: *Harmonic loads, Variable subgrade, Thin beam, Simply supported, Axial force.*

1. INTRODUCTION

Intellectual curiosity on the study of dynamic response of elastic structures to moving loads has been a research interest for over a century. Thus, the complex practical problems usually experienced in railway and highway bridges due to the passage of vehicles and trains of different mass is of technological importance to applied Mathematicians and structural engineers.

The problem of beams resting on elastic foundation especially when subjected to harmonic force moving along a simply supported beam was first addressed by Timoshenko (1922) [1]. This impressive work was later presented in details by Inglis (1967) [2] and later by Qijian and Jianjun (2013) [3].

Several authors have worked on the dynamics of structures on elastic subgrade [4, 5, 6, 7]. Sun (2001) used Fourier transform to solve the problem of steady state response of a beam on viscoelastic foundation subjected to a harmonic line load [8]. The solution is constructed in the form of the convolution of the Green function of the beam. All the different combinations of damping and vibration frequency are discussed and analytical solutions are presented. Omolofe and Oseni (2008) considered the problems of assessing the dynamic response of pre-stressed structural damped uniform beam with rotatory inertia correction factor to concentrated forces with constant velocity [9]. The theoretical analysis of a beam-moving mass system under moving concentrated load was presented by Oni and Omolofe (2005) [10]. They developed an approach that involved generalized Galerkin's method, the Strubles asymptotic technique, integral transformation method and the application of the Fresnel functions. They obtained analytical solutions of the fourth order partial differential equations describing the motion of the non-uniform Rayleigh beams and the condition under which the beam-mass system will experience resonance phenomenon was established.

In a recent development, Ogunyebi et al (2015) presented the dynamic analysis of non-prestressed thin beam on exponential elastic foundation [11]. The effect of foundation rigidity on the deflection of the thin beam was investigated.

This present work focuses on the analytical solution of a uniform prestressed thin beam on variable elastic subgrade subjected to a distributed harmonic loading. The analysis shows the dynamic effect of variable elastic foundation, axial force and damping on the structure as presented in the plots.

2. PROBLEM FORMULATION

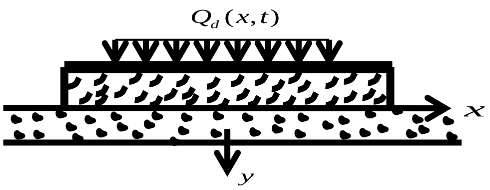

Figure 1: The figure presents the Bernoulli-Euler Beam on variable elastic foundation subjected to harmonic variable loads.

From figure 1 above, denote $y(x,t)$ as the displacement of the Beam in the y direction and x represents the direction of the pavement structure and t the time. The governing fourth order partial differential equation of the Beam is given as

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\mu\frac{\partial^2 y(x,t)}{\partial t^2}-N\frac{\partial^2 y(x,t)}{\partial x^2}+2\mu Z_m\frac{\partial y(x,t)}{\partial t}+F_d(x)y(x,t)=Q_d(x,t) \qquad (2.1)$$

where EI is the rigidity of the Beam, E is the Young's modulus of elasticity, I is the moment of inertia of the Beam, μ is the unit mass of the Beam, Z_m is the damping constant and N is the axial force, $y(x,t)$ is the transverse displacement.

The Beam is assumed to be continuously supported by variable elastic foundation defined by [12]

$$F_d(x)=F_{do}(12-13x+6x^2-x^3) \qquad (2.2)$$

where F_{do} is the elastic foundation constant.

$Q_d(x,t)$ is the external load and for the variable magnitude loading case,

$$Q_d(x,t)=Q_o\cos\omega t H(x-ct) \qquad (2.3)$$

where $H(x-ct)$ is the Heaviside function.

In this work, the boundary condition of the structure is arbitrary and the initial condition without any loss of generality is taken as

$$y(x,0)=0=\frac{\partial y(x,0)}{\partial x} \qquad (2.4)$$

Substituting equations (2.2) and (2.3) into equation (2.1), one obtains a non-homogeneous partial differential equations with variable coefficients

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\mu\frac{\partial^2 y(x,t)}{\partial t^2}-N\frac{\partial^2 y(x,t)}{\partial x^2}+2\mu Z_m\frac{\partial y(x,t)}{\partial t}+F_{do}(12-13x+6x^2-x^3)y(x,t)=Q_o Cos\omega H(x-ct) \qquad (2.5)$$

2.1 Application of Galerkin's transformation:

Assuming harmonic vibration in equation (2.1), the lateral displacement function $y(x,t)$ can be written in the form

$$y(x,t)=V_m(t)\psi(x) \qquad (3.1)$$

where $V_m(t)$ is the coordinate in modal space and $\psi(x)$ is the normal modes of vibration, in fact

$$\psi(x)=\sin\frac{m\pi x}{L} \qquad (3.2)$$

for simply supported end condition.

The application of equation (3.1) in equation (2.5) after further simplification gives

$$\ddot{V}_m(t)R_{11}(m,k)+\dot{V}_m(t)R_{22}(m,k)+\left(R_{33}(m,k)-R_{44}(m,k)+R_{44}(m,k)\right)V_m(t) = R_{66}Cos\omega t\left(\frac{Cosm\pi x}{L}-Cosm\pi\right) \qquad (3.3)$$

where

$$R_{11}(m,k)=\frac{L}{2},\qquad R_{22}(m,k)=W_bL,\qquad R_{33}(m,k)=\frac{m^4\pi^4 EI}{2\mu L},$$
$$R_{44}(m,k)=\frac{m^2\pi^2}{2\mu L},\qquad R_{55}(m,k)=\frac{F_{do}(12-13x+6x^2-x^3)}{2\mu L},\quad R_{66}(m,k)=\frac{MgL}{\mu} \qquad (3.4)$$

Equation (3.3) is a second order ordinary differential equation, subjecting it to Laplace transform, one obtains

$$V_m(s)=\Omega_{m1}\left(\frac{S}{S^2+\omega^2}-\frac{S}{S^2+m_o{}^2}\right)-\Omega_{m2}\left(\frac{S}{S^2+\omega^2}\right)\cdot\frac{1}{(S-\alpha)(S-\beta)} \qquad (3.5)$$

where

$$m_o=\frac{m\pi ct}{L},\qquad \Omega_{m1}=R_{m3},\qquad \Omega_{m2}=R_{m3}(-1) \text{ and } R_{m3}=\frac{R_{66}}{R_{11}(m,k)} \qquad (3.6)$$

and

$$\alpha = \frac{-R_{m1} + \sqrt{R_{m1}^2 - 4R_{m2}}}{2} \quad (3.7)$$

$$\beta = \frac{-R_{m1} - \sqrt{R_{m1}^2 - 4R_{m2}}}{2} \quad (3.8)$$

where

$$R_{m2} = \frac{R_{33}(m,k) - R_{44}(m,k) + R_{55}(m,k)}{R_{11}(m,k)} \quad (3.9)$$

Further simplification of equation (3.5) gives Laplace inversion written as

$$V_m(s) = \frac{1}{\beta - \alpha}\left[\left(g_1(s) f_2(s) - g_1(s) f_1(s) - g_2(s) f_2(s) - g_2(s) f_2(s)\right)\Omega_{m1} - \left(g_1(s) f_2(s) - g_1(s) f_1(s)\right)\Omega_{m2}\right] \quad (3.10)$$

where

$$g_1(s) f_2(s) = \int_0^t e^{\beta(t-u)} Cos\omega u du \quad (3.11)$$

$$g_1(s) f_1(s) = \int_0^t e^{\alpha(t-u)} Cos\omega u du \quad (3.12)$$

$$g_2(s) f_2(s) = \int_0^t e^{\beta(t-u)} Cosm_o u du \quad (3.13)$$

$$g_2(s) f_1(s) = \int_0^t e^{\alpha(t-u)} Cosm_o u du \quad (3.14)$$

and when equation (3.10) is solved via convolution theorem, one obtains

$$V_m(t) = \frac{1}{\beta - \alpha}\Omega_{m1}\frac{\left(\omega Sin\omega t - \beta Cos\omega t + \beta e^{\beta t}\right)}{\omega^2 + \beta^2} - \Omega_{m1}\frac{\left(\omega Sin\omega t - \alpha Cos\omega t + \alpha e^{\alpha t}\right)}{\omega^2 + \alpha^2}$$
$$-\Omega_{m1}\frac{\left(m_o Sinm_o t - \beta Cosm_o t + \beta e^{\beta t}\right)}{m_o^2 + \beta^2} - \Omega_{m1}\frac{\left(m_o Sinm_o t - \alpha Cosm_o t + \alpha e^{\alpha t}\right)}{m_o^2 + \alpha^2}$$
$$-\Omega_{m2}\frac{\left(\omega Sin\omega t - \beta Cos\omega t + \beta e^{\beta t}\right)}{\omega^2 + \beta^2} - \Omega_{m2}\frac{\left(\omega Sin\omega t - \alpha Cos\omega t + \alpha e^{\alpha t}\right)}{\omega^2 + \alpha^2} \quad (3.15)$$

and on inversion yields

$$y(x,t) = \frac{1}{\beta - \alpha}\sum_{m=1}^{n}\left[\Omega_{m1}\frac{\left(\omega Sin\omega t - \beta Cos\omega t + \beta e^{\beta t}\right)}{\omega^2 + \beta^2} - \Omega_{m1}\frac{\left(\omega Sin\omega t - \alpha Cos\omega t + \alpha e^{\alpha t}\right)}{\omega^2 + \alpha^2}\right.$$
$$-\Omega_{m1}\frac{\left(m_o Sinm_o t - \beta Cosm_o t + \beta e^{\beta t}\right)}{m_o^2 + \beta^2} - \Omega_{m1}\frac{\left(m_o Sinm_o t - \alpha Cosm_o t + \alpha e^{\alpha t}\right)}{m_o^2 + \alpha^2}$$
$$\left.-\Omega_{m2}\frac{\left(\omega Sin\omega t - \beta Cos\omega t + \beta e^{\beta t}\right)}{\omega^2 + \beta^2} - \Omega_{m2}\frac{\left(\omega Sin\omega t - \alpha Cos\omega t + \alpha e^{\alpha t}\right)}{\omega^2 + \alpha^2}\right] \cdot Sin\frac{m\pi x}{L} \quad (3.16)$$

which represents the displacement response of the beam on variable elastic foundation subjected to harmonically varying distributed loads on constant velocity.

3. Numerical results and conclusion:

The numerical calculations have been carried out for thin Beam subjected to harmonically varying loading condition. The uniform load of length 12.20m is studied. The velocity of the moving distributed load is taken to be 8.12m/s. the mass per unit length of the structure is 2758.291kg/m. Defining $\omega = 2.05$, P=8407.27 and the values of foundation modulus F_d are varied between 0N/ m^3 and 2000000N/ m^3.

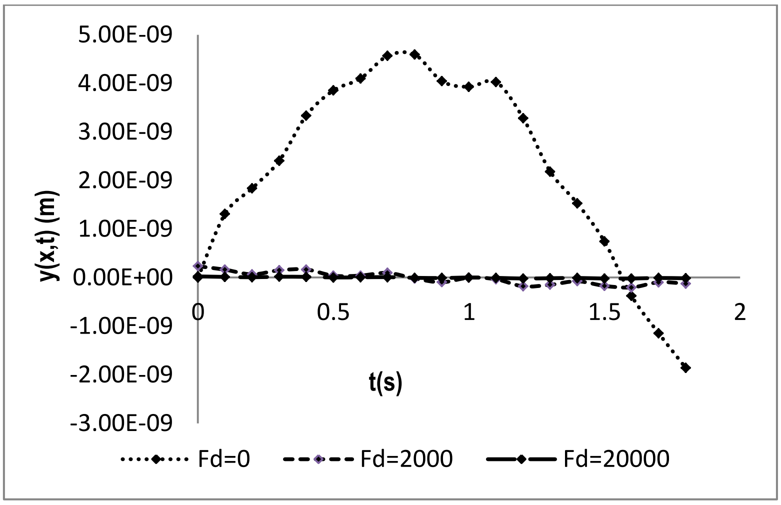

Figure 1: The figure presents the profile of variable elastic foundation effect on dynamic deflection of thin Beam.

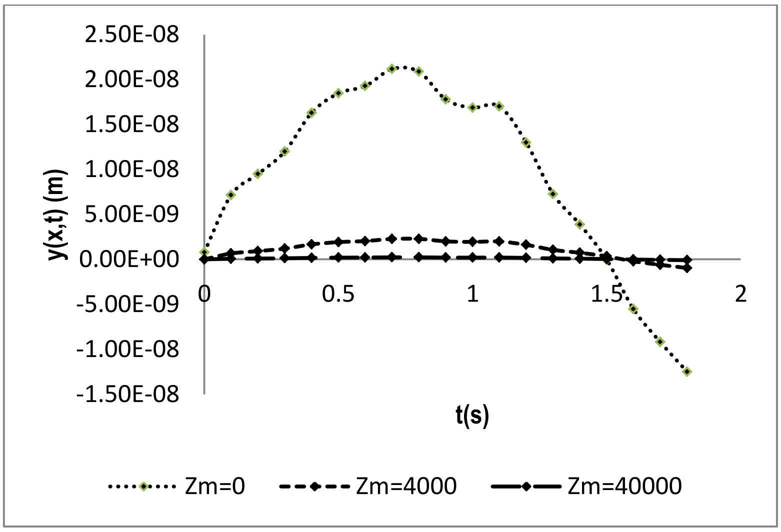

Figure 2: The figure presents the profile of damping effect on dynamic deflection of thin Beam.

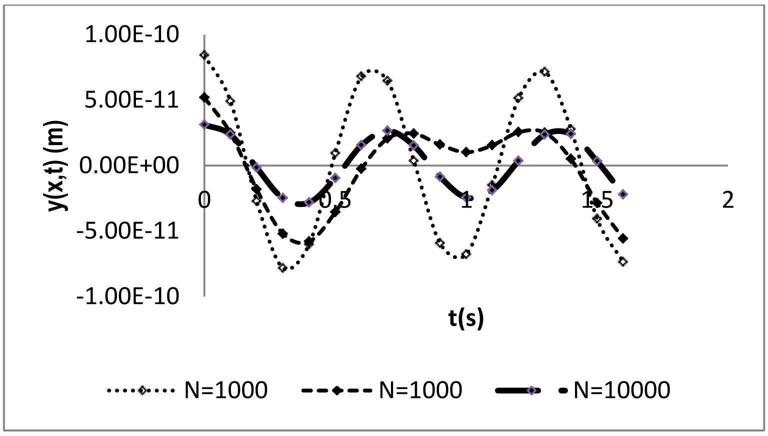

Figure 3: The figure presents the profile of axial force effect on the dynamic deflection of thin Beam**.**

Figure 1 depicts the deflection profile of the effect of variable elastic foundation on thin beam and traversed by harmonic varying distributed masses.

Figure 2 shows the response curve of the effects of damping on the dynamic behavior of the structural element when traversed by harmonic varying distributed masses.

Figure 3 displays the dynamic effect of axial force on the thin Beam resting on variable elastic foundation and traversed by harmonic varying distributed masses. In all the parameters considered, as the value of the foundation modulli F_d, damping N and axial force Z_m increases, as the response amplitude decreases

4. CONCLUSION

In this paper, the response of beam resting on a variable elastic foundation is investigated. The solution of the derived governing equations was effectively achieved using the generalized Garlerkin's method, Laplace transforms and Convolution theory. From the plots, it was discovered that the higher the values of the beam parameters namely, variable elastic foundation F_d, damping Z_m, and axial force N on the simply supported thin beam, lower the response amplitude of the structure when under moving distributed loads at constant speed.
From the analytical and numerical solutions, the following deductions are made.

- As the variable elastic foundation F_d increases, the response amplitudes of thin beam under the action of harmonic varying distributed loads with constant velocity decreases.
- As the damping Z_m increases, the response amplitudes of simply supported beam under the action of harmonic varying distributed loads with constant velocity decreases.
- As the axial force N increases, the response amplitudes of thin beam under the action of moving distributed harmonic loading condition with constant velocity decreases.

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