About the use of FreeFEM++ to solve a diffusion model by domain decomposition methods.

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Abstract. Domain decomposition methods have been particularly developed in recent years with the arrival of parallel computers, and currently represent a very active research direction in numerical analysis. These methods are an essential choice for solving problems on complex geometries, are very effective for large-scale systems, and are also used in the study of coupled models such as the Stokes-Darcy problem. In this work, we present our fundings on domain decomposition methods carried . A series of numerical tests using the FreeFem software were performed and demonstrated the robustness of these methods. The second part concerns a coupled Stokes-Darcy problem. The mathematical formulation as well as the finite element discretization of this problem are presented.

1 Introduction

The first domain decomposition method was developed in the late 19th century by mathematician H.A. Schwarz. The goal was to develop an analytical tool that could extend the Dirichlet principle to more complex domains. The most famous domain in this context is given by the union of a rectangle and a disk, as illustrated in the figure. The Schwarz method requires an overlap between the subdomains. The disadvantage of

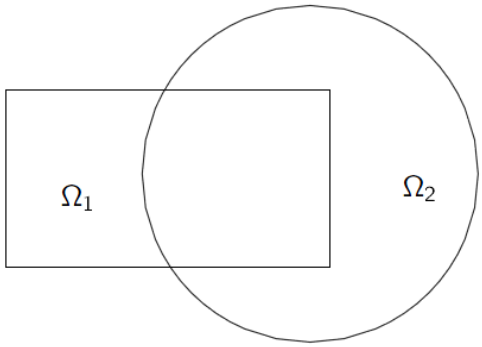

Fig. 1: Domaine original de Schwarz

this type of partitioning is the complexity of its numerical implementation, especially in the case of 3D problems. Moreover, it is difficult to define the overlapping regions for highly complex geometries. Additionally, the convergence in this case is very slow. Non-overlapping methods are the most commonly used nowadays and are the ones we will adopt in the following. They allow reducing the global problem to a problem on the interface.

The rest of the work is outlined as follows: The first section presents the mathematical foundation of the non-overlapping domain decomposition method applied to an

elliptic problem on a domain Ω subdivided into two subdomains Ω_1 and Ω_2 separated by an interface Γ. The variational formulation is given for both the global problem and the subdomain problems. The finite element discretization is also presented, and the LU factorization is used to obtain a condensed system at the interface called the Schur complement. The algebraic version of the Schur complement, called the Steklov-Poincaré operator, is also applied to obtain the same condensed system at the interface. Two iterative algorithms, Dirichlet-Neumann and Neumann-Neumann, are presented for solving the system at the interface. It is shown that these algorithms are preconditioned Richardson methods.

In the second section, some numerical results are presented concerning the convergence and error of domain decomposition methods.

The last section is devoted to the mathematical study of the coupling of a fluid flow governed by the Stokes equation and a flow in porous media governed by the Darcy law. The mathematical model for both the Stokes equation and the Darcy model, as well as the interface conditions between the two zones, are presented. A variational formulation is given, as well as the finite element discretization. Finally, a reduction of the problem to a condensed problem at the interface is presented, which is the Schur complement for our coupled Stokes-Darcy problem.

2 Non-overlapping domain decomposition for the Poisson equation

The objective of this section is to study the mathematical foundations of non-overlapping domain decomposition methods. We mainly consider the elliptic boundary problem of Poisson.

Let Ω be a domain in $\mathbb{R}^d$ (where $d = 2, 3$) with a continuous and Lipschitz boundary $\partial\Omega$. Suppose that Ω is divided into two subdomains Ω_1 and Ω_2 with an interface Γ, such that $\bar{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, $\Omega_1 \cap \Omega_2 = \varnothing$, and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. Figure 2 illustrates a rectangular domain Ω divided into two subdomains.

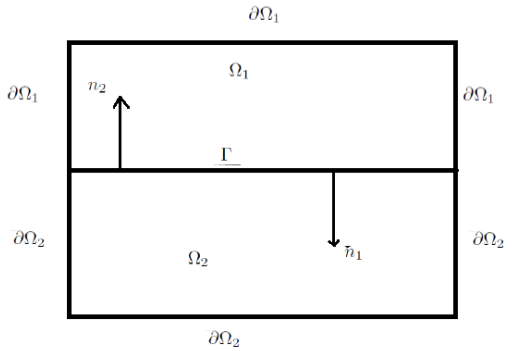

Fig. 2: Décomposition en deux sous-domaines

For a heterogeneous medium, we consider the following elliptic problem:

$$\begin{cases} -\nabla(\mu\nabla u) = f, dans \ \ \Omega, \\ u = 0, \qquad\qquad sur \ \ \partial\Omega. \end{cases} \tag{1}$$

such that $\mu \in \mathbb{L}^\infty(\Omega)$, $\inf\limits_{\Omega} \mu > 0$ et $f \in \mathbb{L}^2(\Omega)$.

The function μ is given by :

$$\mu(x) = \begin{cases} \mu_1(x), dans \ \ \Omega_1, \\ \mu_2(x), dans \ \ \Omega_2, \end{cases}$$

with $\mu_i \in \mathbb{L}^\infty(\Omega_i)$ i=1,2.

To simplify, the theoretical results will be presented for the case $\mu = 1$, which corresponds to the following Poisson problem:

$$\begin{cases} -\Delta u = f, dans \ \ \Omega, \\ u = 0, \qquad sur \ \ \partial\Omega \end{cases} \tag{2}$$

The results obtained for the Poisson problem (2) remain valid for the heterogeneous problem (1) and vice versa.

We consider the following local problems:

$$\begin{cases} -\Delta u_1 = f_1, dans \ \ \Omega_1, \\ u_1 = 0, \qquad sur \ \ \partial\Omega_1 \backslash \Gamma \end{cases} \tag{3}$$

$$\begin{cases} -\Delta u_2 = f_2, dans \ \ \Omega_2, \\ u_2 = 0, \qquad sur \ \ \partial\Omega_2 \backslash \Gamma \end{cases} \tag{4}$$

où $f_1 = f_{|\Omega_1}$ et $f_2 = f_{|\Omega_2}$

with the following boundary conditions:

– Condition of admissibility in the interface

$$u_1 = u_2 \quad sur \quad \Gamma. \tag{5}$$

– Equilibrium condition

$$\frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2} \quad sur \quad \Gamma. \tag{6}$$

with $u_i = u_{|\Omega_i}$, $i = 1, 2$ is the restriction of u on Ω_i et n_i is the outward normal of the domain Ω_i. The domain decomposition methods consist of reformulating the problem under study into an equivalent problem whose unknowns are functions defined on the interface of the subdomains. The Schur complement methods (primal and dual) will be studied for the problem (2)-(6). The Steklov-Poincaré operator, which is an algebraic version of the Schur complement, will also be presented.

3 Weak formulation

In this section, we provide the weak form of the global problem and local problems. We then show that these problems are well-posed, have unique solutions, and that solving the global problem is equivalent to solving the two local problems with the two interface conditions. For the variational formulation of the global problem, we consider the following Hilbert spaces: $H^1(\Omega)$:=$\{\mathrm{v} \in \mathbb{L}^2(\Omega) / D_j \ \mathrm{v} \in \mathbb{L}^2(\Omega)\ ,j = 0, ...d\}$ et $H_0^1(\Omega) := \{v \in H^1(\Omega) / v|_{\partial\Omega} = 0\}$. The space $H^1(\Omega)$ has the following norm $\| v \|_{H^1}$:= $(\| v \|_{\mathbb{L}^2}^2 + \| \nabla v \|_{\mathbb{L}^2}^2)^{\frac{1}{2}}$ with $\| v \|_{\mathbb{L}^2}$:= $(v, v)^{\frac{1}{2}}$ et $(u, v) = \int_\Omega uvdx$.

Let us denote: $V := H_0^1(\Omega)$. Let u be a fairly regular solution of (2) and $v \in V$. Multiplying the first equation of (2) by v and integrating over Ω we obtain:

$$int_\Omega - (\Delta u)vdx = \int_\Omega fvdx.$$

By the formula of Green we will have then,

$$int_\Omega \nabla u.\nabla vdx - \int_{\partial\Omega} \partial_n uvd\sigma = \int_\Omega fvdx.$$

Now $v = 0$ on the edge, the term $int_{\partial\Omega}\partial_n uvd\sigma$ cancels out, so we get the following weak form:

$$\text{trouver } u \in V \quad a(u, v) = (f, v) \quad v \in V \tag{7}$$

with $(f, v) = \int_\Omega fvdx$ et $a(u, v) := \int_\Omega \nabla u \cdot \nabla v$

Theorem 3.1. *The problem (7) admits a unique solution.*

The proof can be done through the Lax-Milligram theorem and the Poincaré inequality (For more details, we can refer to [5]).

3.1 Variational formulation of local problems : Trace operator and extension operator

The trace space of $H^1(\Omega)$ on the edge $\partial\Omega$ is denoted $H^{1/2}(\partial\Omega)$. For a non empty open $\Sigma \subset \partial\Omega$ the trace space will be denoted $H^{1/2}(\Sigma)$. The trace operator

$$\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$$

$$v \longmapsto \gamma_0(v) = v_{|_{partial\Omega}}$$

is surjective and continuous and we have the following trace inequality: There exists a constant $C_\Omega > 0$ such that

$$\| v_{|_{\partial\Omega}} \|_{H^{1/2}(\partial\Omega)} \leq C_\Omega ||v||_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

Let us consider, $\Lambda := \{\eta \in H^{1/2}(\Gamma)/\eta = v_{|_\Gamma} \quad pour \quad v \in V\}$. The space Λ has the norm $\| \eta \|_\Lambda := \inf_{v \in V_i, v_{|\Gamma} = \eta} \| v \|_{\mathbb{H}^1(\Omega_i)}$. Let the follosing Hilbert spaces $V_i := \{v_i \in H^1(\Omega_i)/v_i|_{\partial\Omega \cap \partial\Omega_i} = 0\}$ i=1,2. The trace operator

$$\gamma_i : V_i \longrightarrow \Lambda$$

is continuous with respect to the norm *parallel.* $\|_\Lambda$, hence the trace inequality follows:

$$\exists C_i^* \quad telle \quad que : \quad \| v_i|_\Gamma \|_\Lambda \leq C_i^* \| v_i \|_{\mathbb{H}^1(\Omega_i)} \quad v_i \in V_i$$

We note by $\Re_i (i = 1, 2)$ the following extension operator:

$$\Re_i : \Lambda \longrightarrow V_i$$

$$\eta \longmapsto \Re_i \eta$$

with $(\Re_i \eta)_{|\Gamma} = \eta$. This operator is continuous and is not unique.

Equivalence between the global problem and the local problems Let us consider: $a_i(u_i, v_i) := (\nabla u_i, \nabla v_i)_{\Omega_i}$ i=1,2, and $V_i^0 := \{v_i \in V_i / v_i|_\Gamma = 0\} = H_0^1(\Omega_i)$ i=1,2. The following lemma shows the equivalence between the weak form of the global problem and the local problems.

Lemma 3.2. *The fish problem (2) is equivalent to:*
Find $u_1 = u_{|\Omega_1}$ *et* $u_2 = u_{|\Omega_2}$, *such that :*

$$(FVP) \begin{cases} a_1(u_1, v_1) = (f_1, v_1), & v_1 \in V_1^0, \\ u_1 = u_2, & sur \ \ \Gamma \\ a_2(u_2, v_2) = (f_2, v_2), & v_2 \in V_1^0, \\ a_2(u_2, \Re_2 \eta) = (f_2, \Re_2 \eta)_{\Omega_2} + (f_1, \Re_1 \eta)_{\Omega_1} - a_1(u_1, \Re_1 \eta), & \forall \eta \in \Lambda \end{cases}$$

Proof. Let u be the solution of (7). We have $u_i = u|\Omega_i \in V_i$ and the first three equations of (FVP) are satisfied. Moreover, for each $\eta \in \Lambda$, we can define the extension operator $\Re$ by:

$$\Re : \Lambda \longrightarrow V$$

$$\eta \longmapsto \Re \eta$$

with

$$\Re \eta := \begin{cases} \Re_1 \eta, \, sur \ \ \Omega_1, \\ \Re_2 \eta, \, sur \ \ \Omega_2 \end{cases}$$

Thus we have $a(u, \Re\eta) = (f, \Re\eta)$, so $\sum i = 1^2 a_i(u_i, \Re_i \eta) = \sum_{i=1}^2 (f_i, \Re_i \eta)\Omega$, and therefore the last equation is satisfied.
Conversely, if u_1 and u_2 are solutions of (FVP), let:

$$u = \begin{cases} u_1, \, sur \ \ \Omega_1, \\ u_2, \, sur \ \ \Omega_2 \end{cases}$$

It is clear that $u \in V$. For each $v \in V$, we have $\eta := v|\Gamma \in \Lambda$, and by definition of $\Re\eta$, we have $(v|\Omega_i - \Re_i\eta) \in V_i^0$. Using equations 1, 3, and 4 of (FVP), we have:

$$\begin{aligned} a(u,v) &= \sum_{i=1}^{2}[a_i(u_i, v|_{\Omega_i} - \Re_i\eta) + a_i(u_i, \Re_i\eta)] \\ &= \sum_{i=1}^{2}[(f, v|_{\Omega_i} - \Re_i\eta)_{\Omega_i} + (f, \Re_i\eta)_{\Omega_i}] \\ &= (f,v) \quad \forall v \in V. \end{aligned}$$

Therefore, u is a solution of (7). □

4 Discretization by Finite Element Method

We will give a finite element discretization of the (FVP) problem. For this, we consider a regular triangulation $\mathcal{T}h$ of the domain $\overline{\Omega} : \overline{\Omega} = \bigcup K \in \mathcal{T}_h K$, where each $K \in \mathcal{T}_h$ is a non-empty interior triangle with $\mathring{K_1} \cap \mathring{K_2} = \emptyset$ for all $K_1, K_2 \in \mathcal{T}h$, and if $e = K_1 \cap K_2 \neq \emptyset$, then e is a face, edge, or vertex shared by K_1 and K_2. We consider the space

$$X_h(\Omega) := v_h \in C^0(\overline{\Omega})/v_h|K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h,$$

where $\mathbb{P}1(K)$ is the space of polynomials on the element K of degree less than or equal to 1, and the space

$$V_h = v_h \in X_h(\Omega)/ \quad v_h|\partial\Omega = 0 = X_h(\Omega) \cap \mathbb{H}_0^1(\Omega),$$

where $\mathbb{H}_0^1(\Omega)$ is the Sobolev space of functions in $\mathbb{H}^1(\Omega)$ with zero trace on the boundary $\partial\Omega$.

We denote $F(v) = (f, v)$, and therefore the problem (7) becomes:

$$\text{trouver} \quad u_h \in V_h : a(u_h, v_h) = F(v). \tag{8}$$

Let ${\varphi_j}_{j=1}^N$ be a basis for the space V_h. Suppose that the interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$ is a union of edges or faces of the triangulation $\mathcal{T}h$. We introduce the following partition of the nodes in the domain: let $x_j^1, 1 \leq j \leq N_1$ be the nodes in the subdomain Ω_1, let $x_j^2, 1 \leq j \leq N_2$ be the nodes in the subdomain Ω_2, and let $x_j^\Gamma, 1 \leq j \leq N\Gamma$ be the nodes on the interface Γ. We also partition the basis functions by denoting φ_j^i the functions associated with nodes $x_j^i \quad (i = 1, 2 \quad j = 1, ...N_i)$ and φ_j^Γ the functions associated with nodes x_j^Γ on the interface.

We take the φ_j^α ($\alpha = 1, 2$ or Γ) as test functions in the problem (8), and thus we have the following problem:

Find $u_h \in V_h$ such that:

$$\begin{cases} a(u_h, \varphi_j^1) = F(\varphi_j^1), \ \forall j = 1...., N_1, \\ a(u_h, \varphi_j^2) = F(\varphi_j^2), \ \forall j = 1...., N_2, \\ a(u_h, \varphi_j^\Gamma) = F(\varphi_j^\Gamma), \forall j = 1...., N_\Gamma, \end{cases} \tag{9}$$

Let $a_i(.,.)$ and $F_i(.)$ be the restrictions of the form $a(.,.)$ and $F(.)$ to the subdomain Ω_i, i=1,2. We consider the space $V_{i,h}$ of polynomial functions of degree less than or equal to 1, which vanish on the boundary $\partial\Omega_i$ except for the interface Γ, and the space $V_{i,h}^0 := v_h \in V_{i,h}/v_h|\Gamma = 0$. Let $u_h^i = u_h|\Omega_i \in V_{i,h}$, then the problem (9) can be written in the following multi-domain formulation:

$$\begin{cases} a_1(u_h^1, \varphi_k^1) = F_1(\varphi_k^1), & \forall k = 1...., N_1, \\ a_2(u_h^2, \varphi_k^2) = F_2(\varphi_k^2), & \forall k = 1...., N_2, \\ a_1(u_h^1, \varphi_k^\Gamma|_{\Omega_1}) + a_2(u_h^2, \varphi_k^\Gamma|_{\Omega_2}) = F_1(\varphi_k^\Gamma|_{\Omega_1}) + F_2(\varphi_k^\Gamma|_{\Omega_2}), & \forall k = 1...., N_\Gamma, \end{cases} \tag{10}$$

We decompose the function u_h on the basis $\{varphi_j\}$ of the space V_h, we have :

$$u_h(x) = \sum_{j=1}^{N_1}(u_j^1\varphi_j^1(x)) + \sum_{j=1}^{N_2}(u_j^2\varphi_j^2(x)) + \sum_{j=1}^{N_\Gamma}(u_j^\Gamma\varphi_j^\Gamma(x)). \tag{11}$$

where $u_j^\alpha := u_h(x_j^\alpha)(j = 1, ..., N_\alpha, \alpha = 1, 2, \Gamma)$ are the coefficients of the linear combination representing u_h in the $\{\varphi_j\}$ basis. In the same way, we also obtain :

$$u_h^i(x) = \sum_{j=1}^{N_i}(u_j^i\varphi_j^i(x)) + \sum_{j=1}^{N_\Gamma}(u_j^\Gamma\varphi_{j|\Omega_i}^\Gamma(x)). \tag{12}$$

where $\varphi_{j|\Omega_i}^\Gamma$ is the restriction of φ_j^Γ on Ω_i.

By substituting (12) in the first two equations of (10), we find

$$\sum_{j=1}^{N_1} u_j^1 a_1(\varphi_j^1(x), \varphi_k^1(x)) + \sum_{j=1}^{N_\Gamma} u_j^\Gamma a_1(\varphi_{j|\Omega_1}^\Gamma(x), \varphi_k^1(x)) = F(\varphi_k^1(x)) \tag{13}$$

and

$$\sum_{j=1}^{N_2} u_j^1 a_2(\varphi_j^2(x), \varphi_k^2(x)) + \sum_{j=1}^{N_\Gamma} u_j^\Gamma a_2(\varphi_{j|\Omega_2}^\Gamma, \varphi_k^2(x)) = F(\varphi_k^2(x)) \tag{14}$$

Let define the following matrix

$$(A_{ii})_{kj} = a_i(\varphi_j^i, \varphi_k^i), \quad i = 1, 2$$

$$(A_{i\Gamma})_{kj} = a_i(\varphi_{j|\Omega_i}^\Gamma, \varphi_k^i), \quad i = 1, 2$$

$$(A_{\Gamma i})_{kj} = a_i(\varphi_j^i, \varphi_{k|\Omega_i}^\Gamma), \quad i = 1, 2$$

$$(A^i_{\Gamma\Gamma})_{kj} = a_i(\varphi^{\Gamma}_{j|_{\Omega_i}}, \varphi^{\Gamma}_{k|_{\Omega_i}}), \quad i = 1, 2$$

$$(b_i)_k = F_i(\varphi^i_k), \quad i = 1, 2$$

$$(b^{\Gamma}_i)_k = F_i(\varphi^{\Gamma}_{k|_{\Omega_i}}), \quad i = 1, 2$$

The system (10) can therefore be written as :

$$\begin{cases} A_{11}u_1 + A_{1\Gamma}u_{\Gamma} = b_1, \\ A_{22}u_2 + A_{2\Gamma}u_{\Gamma} = b_2, \\ A_{\Gamma 1}u_1 + A_{\Gamma 2}u_2 + (A^1_{\Gamma\Gamma} + A^2_{\Gamma\Gamma})u_{\Gamma} = b^{\Gamma}_1 + b^{\Gamma}_2, \end{cases} \tag{15}$$

où $u_1 = (u^1_j)_{j=1\ldots N_1}$, $u_2 = (u^2_j)_{j=1\ldots N_2}$, $u_{\Gamma} = (u^{\Gamma}_j)_{j=1\ldots N_{\Gamma}}$, $b_1 = (b_j)_{j=1\ldots N_1}$, $b_2 = (b_j)_{j=1\ldots N_2}$, $b^{\Gamma}_1 = (b^{\Gamma}_{1j})_{j=1\ldots N_1}$, $b^{\Gamma}_2 = (b^{\Gamma}_{2j})_{j=1\ldots N_2}$ Then we have the system in the following matrix form:

$$\begin{pmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_{\Gamma} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_{\Gamma} \end{pmatrix} \tag{16}$$

with $A_{\Gamma\Gamma} = A^1_{\Gamma\Gamma} + A^2_{\Gamma\Gamma} \quad et \quad b_{\Gamma} = b^1_{\Gamma} + b^2_{\Gamma}$.

5 Method of primal Schur's complement

The system (16) can be written in the following form:

$$\begin{cases} A_{ii}u_i + A_{i\Gamma}u_{\Gamma} = b_i, i = 1, 2 \\ A_{\Gamma 1}u_1 + A_{\Gamma 2}u_2 + (A^1_{\Gamma\Gamma} + A^2_{\Gamma\Gamma})u_{\Gamma} = b_{\Gamma}, \end{cases} \tag{17}$$

It is assumed that the blocks A_{ii} admit a factorization LU, the first two equations of the system (17) then becomes :

$$u_i = A^{-1}_{ii}(b_i - A_{i\Gamma}u_{\Gamma}) \quad i = 1, 2 \tag{18}$$

We replace $u_i, i = 1, 2$ in the third equation, so we have the condensed system at the interface

$$S_{\Gamma\Gamma}u_{\Gamma} = C_{\Gamma} \tag{19}$$

with $S_{\Gamma\Gamma} = A_{\Gamma\Gamma} - A_{\Gamma 1}A^{-1}_{11}A_{1\Gamma} - A_{\Gamma 2}A^{-1}_{22}A_{2\Gamma}$ Schur's complement to the unknowns on Γ, and $C_{\Gamma} = (b^{(1)}_1 - A_{\Gamma 1}A^{-1}_{11}b_1) + (b^{(2)}_{\Gamma} - A_{\Gamma 2}A^{-1}_{22}b_2) = C^{(1)}_{\Gamma} + C^{(2)}_{\Gamma}$.

The solution of the system (19) can be done by an iterative method. The corresponding S_{Γ} matrix has a better conditioning compared to the starting matrix A, and thus allows a better convergence speed (see [1] page 98).

6 Steklov-Poincaré operator

V. A. Steklov 1864 – 1926

H. Poincaré 1854 – 1912

Let $\eta := u_{|\Gamma}$ be the restriction of the solution u of (2) on Γ. Consider the following Dirichlet problem:

$$\begin{cases} -\Delta(w_i) = f, dans \ \ \Omega_i, \\ w_i = 0, \qquad sur \ \ \partial\Omega_i|_\Gamma, \quad i = 1, 2 \\ w_i = \eta, \qquad sur \ \ \Gamma \end{cases} \tag{20}$$

The solution w_i of (20) can be written as

$$w_i = u_i^0 + u_i^*. \tag{21}$$

The solution w_i of (20) can be written in the following form where u_i^0 and u_i^* are the solutions of the following Dirichlet problems:

$$\begin{cases} -\Delta(u_i^0) = 0, dans \ \ \Omega_i, \\ u_i^0 = 0, \qquad sur \ \ \partial\Omega_i|_\Gamma, \quad i = 1, 2 \\ u_i^0 = \eta, \qquad sur \ \ \Gamma \end{cases} \tag{22}$$

and

$$\begin{cases} -\Delta(u_i^*) = f, dans \ \ \Omega_i, \\ u_i^0 = 0, \qquad sur \ \ \partial\Omega_i|_\Gamma, \quad i = 1, 2 \\ u_i^0 = 0, \qquad sur \ \ \Gamma \end{cases} \tag{23}$$

The solution u_i^0 of (22) is called the harmonic extension of η to Ω_i and will be denoted $H_i\eta$, while the solution u_i^* of (23) will be denoted $G_i f$ and will be called the resolvent operator. We have then

$$w_i = H_i\eta + G_i f. \tag{24}$$

Remark 6.1. *$G_i f$ depends only on f, while $H_i\eta$ depends only on the value of the solution u on Γ.*

The solution w_i of (24) is equal to u_i if and only if we have the condition $frac\partial w_1 \partial n = \frac{\partial w_2}{\partial n}$ on Γ, where n is the normal on $\Gamma (n = n_1 = -n_2)$. And since we have $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$, then we will have $(\frac{\partial H_1 \eta + \partial G_1 f}{\partial n}) = (\frac{\partial H_2 \eta + \partial G_2 f}{\partial n})$ therefore

$$(\frac{\partial H_1 - \partial H_2}{\partial n})\eta = (\frac{\partial G_2 f - \partial G_1 f}{\partial n}) \tag{25}$$

Let χ be the function defined as follows:

$$\begin{aligned} \chi &= \frac{\partial G_2 f}{\partial n} - \frac{\partial G_1 f}{\partial n} \\ &= -\frac{\partial G_2 f}{\partial n_2} - \frac{\partial G_1 f}{\partial n_1} \\ &= -\sum_{i=1}^{2} (\frac{\partial G_i f}{\partial n_i}) \end{aligned}$$

This function depends only on f and the normal on Γ. Soit $\mathbb{S}$ l'opérateur de Steklov-Poincaré définit par :

$$\begin{aligned} \mathbb{S}\eta &= \frac{\partial H_1 \eta}{\partial n_1} - \frac{\partial H_2 \eta}{\partial n_2} \\ &= \sum_{i=1}^{2} (\frac{\partial H_i \eta}{\partial n_i}) \\ &= \mathbb{S}_1 + \mathbb{S}_2. \end{aligned}$$

The equation (25) then becomes the Steklov-Poincaré equation on the following Γ interface:

$$\mathbb{S}\eta = \chi \quad sur \quad \Gamma. \tag{26}$$

7 Iterative methods for domain decomposition: Dirichlet-Neumann algorithm

The Dirichelt-Neumann method consists in solving a Dirichlet problem in the first domain, with a Dirichlet data λ^k on Γ , and then solving a mixed Dirichlet-Neumann problem on the second domain where we use the value of the flux of the previous solution on Γ as a Neumann condition, and a homogeneous condition on the remainder of the domain edge. The algorithm for the heterogeneous problem (1) is given as follows: • λ^0 given

• for each iteration $k \succeq 0$

– solve

$$\begin{cases} -\nabla(\mu_1 \nabla u_1^{k+1}) = f, dans \ \ \Omega_1, \\ u_1^{k+1} = 0, \qquad\qquad sur \ \ \partial\Omega_1|_\Gamma, \\ u_1^{k+1} = \lambda^k, \qquad\qquad sur \ \ \Gamma \end{cases} \tag{27}$$

– solve :

$$\begin{cases} -\nabla(\mu_2 \nabla u_2^{k+1}) = f, dans \ \ \Omega_2, \\ u_2^{k+1} = 0, \qquad\qquad sur \ \ \partial\Omega_2|_\Gamma, \\ \mu_2 \frac{\partial u_2^{k+1}}{\partial n} = \mu_1 \frac{\partial u_1^{k+1}}{\partial n}, \ sur \ \ \Gamma \end{cases} \tag{28}$$

– update λ^{k+1}:

$$\lambda^{k+1} = \theta u_2^{k+1} + (1-\theta)\lambda^k. \tag{29}$$

où θ is a relaxation parameter to accelerate the convergence.

• Stop criterion:
$\|\lambda^{k+1} - \lambda^k\| \leq \epsilon$, avec ϵ une tolérance donnée.

The following theorem shows that the Dirichlet-Neumann method is nothing else than the preconditioned Richardson method.

Theorem 7.1. *The Dirichlet-Neumann method is equivalent to the preconditioned Richardson method with preconditioner S_2 applied to the Steklov-Poincaré interface equation $S\lambda = \chi$ i.e. we have :*

$$\lambda^{k+1} = \lambda^k + \theta S_2^{-1}(\chi - S\lambda^k).$$

7.1 Neumann-Neumann Algorithm

For Neumann-Neumann we first solve a Dirichlet problem in each subdomain with Dirichlet data λ^k on Γ, then two Neumann problems where we use the difference of the fluxes of the previous solutions as Neumann data on Γ. We have the following algorithm to solve the heterogeneous problem (1): • λ^0 donnée sur Γ.

• for each $k \succeq 0$,

– Pour i=1,2 solve :

$$\begin{cases} -\nabla(\mu_i \nabla u_i^{k+1}) = f, dans \ \ \Omega_i, \\ u_i^{k+1} = 0, \qquad\qquad sur \ \ \partial\Omega_i|_\Gamma, \\ u_i^{k+1} = \lambda^k, \qquad\qquad sur \ \ \Gamma \end{cases} \tag{30}$$

– For i=1,2 solve :

$$\begin{cases} -\nabla(\mu_i \nabla \psi_i^{k+1}) = 0, \qquad\qquad dans \ \ \Omega_i, \\ \psi_i^{k+1} = 0, \qquad\qquad sur \ \ \partial\Omega_i|_\Gamma, \\ \mu_i \frac{\partial \psi_i^{k+1}}{\partial n} = \mu_1 \frac{\partial u_1^{k+1}}{\partial n} - \mu_2 \frac{\partial u_2^{k+1}}{\partial n}, sur \ \ \Gamma \end{cases} \tag{31}$$

– mettre à jour λ^{k+1} :

$$\lambda^{k+1} = \lambda^k - \theta(\sigma_1 \psi_{1|\Gamma}^{k+1} - \sigma_2 \psi_{2|\Gamma}^{k+1}). \tag{32}$$

• Stop criterion :
$\|\lambda^{k+1} - \lambda^k\| \leq \epsilon$, avec ϵ une tolérance donnée.

The parameter θ is a relaxation parameter to speed up the convergence. The positive weights σ_1 and σ_2 allow to weight the contributions of the flows following the Ω_1 and Ω_2 domains to attenuate the effects of the jumps of the diffusion functions μ_1 and μ_2 for example. In general we will take $\sigma_1 + \sigma_2 = 1$ and in the case of constant diffusion coefficients by subdomain, we take $\sigma_i = \frac{\mu_i}{\mu_1+\mu_2}$.

Theorem 7.2. *The Neumann-Neumann method is equivalent to the preconditioned Richardson method with preconditioner* $N := (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$ *applied to the Steklov-Poincaré interface equation* $S\lambda = \chi$ *c'est à dire qu'on a :*

$$\lambda^{k+1} = \lambda^k + \theta(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})(\chi - S\lambda^k)$$

8 Numerical results: Diffusion problem

FreeFem++ is a free software originally developed by Frédéric Hecht, researcher at the Jacques-Louis Lions Laboratory of the Pierre et Marie Curie University in Paris. It allows to numerically solve Partial Differential Equations (PDE) by the Finite Element Method (FEM). The FreeFEM++ language allows to quickly specify $(2D, 3D)$ PDEs, to manipulate several $(2D, 3D)$ meshes and to write scripts in C++ to define numerical algorithms for problems: nonlinear, coupled...

FreeFEM++ allows to create meshes of the studied domains. Figures (3) and (4) show examples of meshes (uniform and non-uniform) created by FreeFEM.

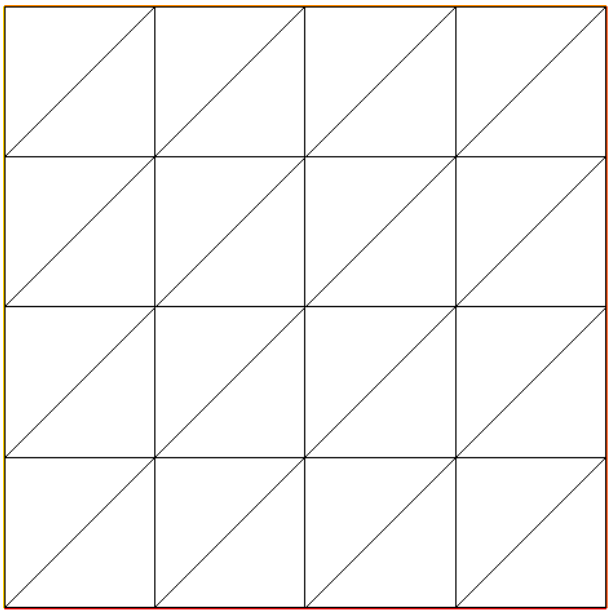

Fig. 3: Maillage uniforme sous FreeFEM

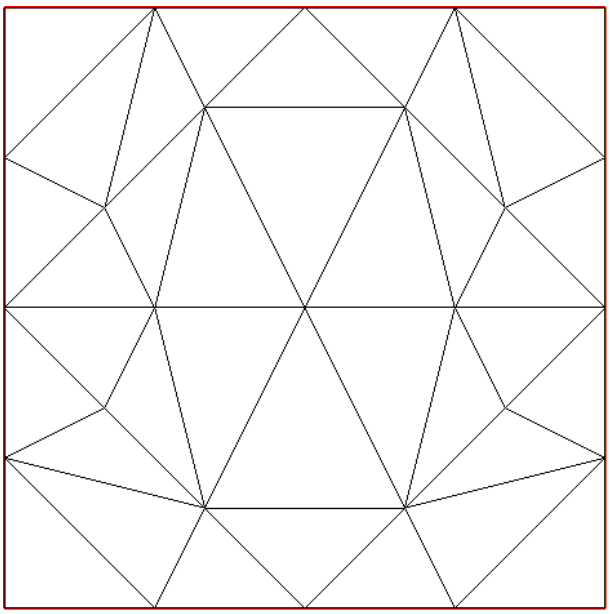

Fig. 4: Maillage non-uniforme sous FreeFEM

We consider the following domain $\Omega = [0, 1] \times [0, 1]$, we fix the number of nodes on each interface of this domain to 5 nodes, then we generate a uniform mesh on the

whole domain. The problem to solve on all Ω is the following:

$$\begin{cases} -\Delta u = 1, \, dans \;\; \Omega, \\ u = 0, \qquad sur \;\; \partial\Omega. \end{cases} \tag{33}$$

The graph (5) representing the solution on the whole domain Ω.

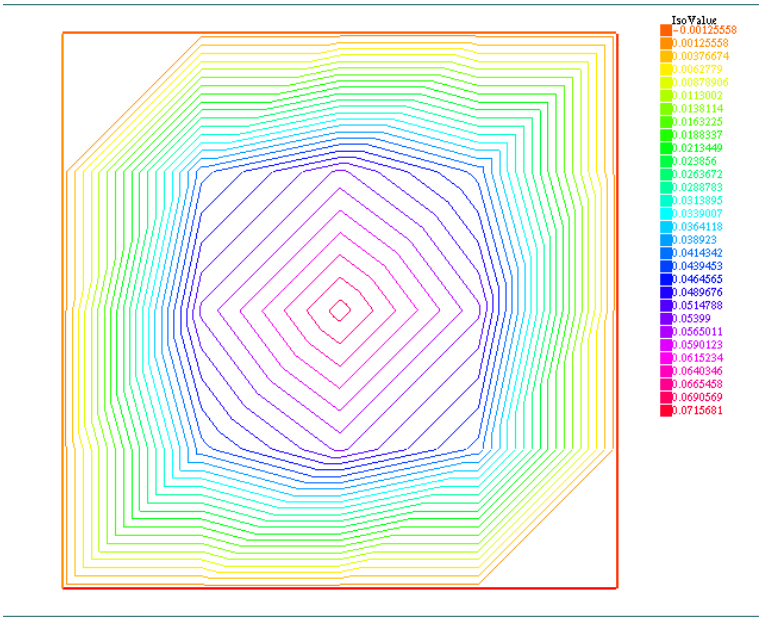

Fig. 5: Solution sur tout le domaine Ω.

We decompose the domain Ω in two subdomains Ω_1 and Ω_2, we solve the problem (33) on each subdomain using one of the Dirichlet-Neumann or Neumann-Neumann algorithms, the figure (??) represents the approximate solution on Ω_1, while the figure (??) represents that on Ω_2.

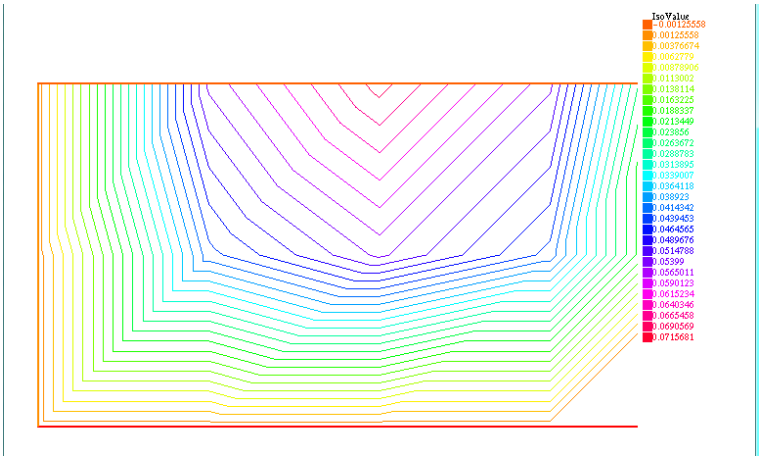

Fig. 6: Solution sur Ω_1

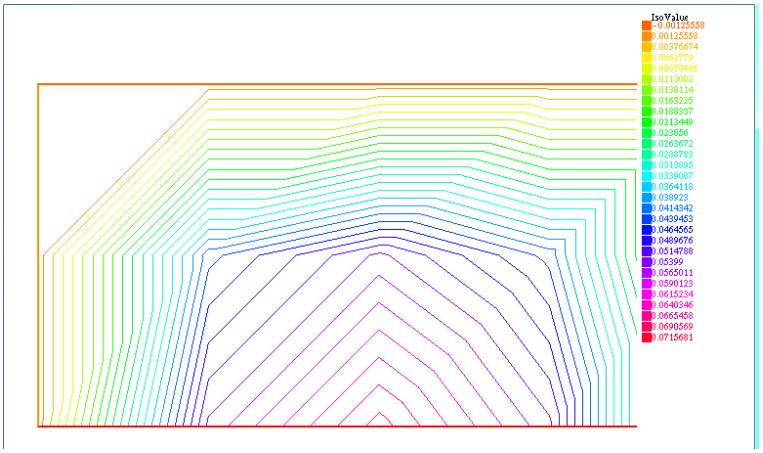

Fig. 7: Solution sur Ω_2

Let u_1 be the solution on the first domain Ω_1, and u_2 that on the second domain Ω_2, we characterize the stopping criterion as follows:

- We set a tolerance ϵ.
- We initialize the error by 0 (ER0=0).
- At each iteration, the following relative error is calculated:

$$ER1 = \frac{||u_1 - u_2||^2_{\mathbb{L}^2}}{||u_1||^2_{\mathbb{L}^2}}. \tag{34}$$

• Case 1 :
Si $|ER1 - ER0| > \epsilon$, we continue, and we reset the calculated error : $ER0 = ER1$.
• Case 2 :
Si $|ER1 - ER0| \leq \epsilon$, we stop the loop.

For the Poisson equation, both Dirichlet-Neumann and Neumann-Neumann algorithms converge in one iteration.

Let $\Omega = [0, 1] \times [0, 1]$ and

$$\begin{cases} -\Delta u = sin(\pi x)sin(\pi y), dans \ \ \Omega, \\ u = 0, \qquad\qquad\qquad\qquad sur \ \ \partial\Omega. \end{cases} \tag{35}$$

The analytical solution of the problem (**??**) is of the form :

$$u(x, y) = \frac{1}{2\pi^2} sin(\pi x)sin(\pi y). \tag{36}$$

We are going to make a comparison between the values taken on the Γ interface between the two domains Ω_1 and Ω_2 by this analytical solution, by the solution without domain decomposition and the solution with domain decomposition. We fix y =0.5 then we vary

x and we recover the values of $u(x, 0.5)$, of the solution without domain decomposition and those with domain decomposition.

Error calculation:

The following graph illustrates the errors $\|u_{ADDM} - u_{exact}\|_{\mathbb{L}^2}$, and $\|u_{SDDM} - u_{exact}\|_{\mathbb{L}^2}$, where u_{ADDM} represents the approximate solution obtained with domain decomposition, u_{SDDM} the approximate solution recovered without domain decomposition, and u_{exact} the exact solution given in (**??**). We always use the criterion (34) as a stopping criterion.

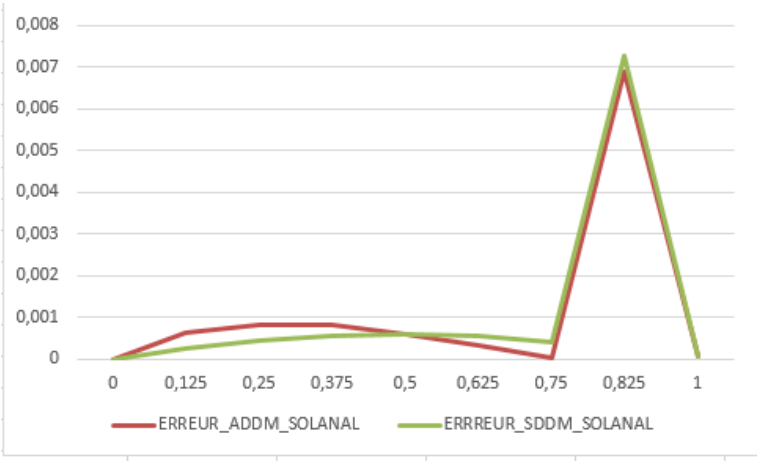

Fig. 8: Error comparison on the interface with the NN algorithm.

The green curve represents the computed error between the analytical solution and the approximated solution without domain decomposition, while the brown curve represents the error between the analytical solution and the approximated solution with domain decomposition. The two curves are not identical, because of the errors between the approximated solution without domain decomposition and the one with domain decomposition on the $(y = 0.5)$ interface.

8.1 Influence of the jumps of the coefficients for a heterogeneous problem

We are interested in the problem:

$$\begin{cases} -\nabla(\mu \nabla u) = 1, \, dans \;\; \Omega = [0,1] \times [0,1], \\ u = 0, \qquad\qquad\quad sur \;\; \partial\Omega \end{cases} \tag{37}$$

with

$$\mu(x) = \begin{cases} \mu_1, \, dans \;\; \Omega_1, \\ \mu_2, \, dans \;\; \Omega_2, \end{cases} \tag{38}$$

For each value of (μ_1, μ_2), we refine the mesh (we vary the value of n), and we recover the number of iterations needed for the convergence using the stopping criterion (34).

Remark 8.1. *The error computed above is done on the interface (y=0.5), while the solution on the interface is recovered in a different way according to the choice of the algorithm.*

The table (1) presents the results for the two algorithms Dirichlet-Neuamnn and Neumann-Neumann.

(μ_1, μ_2)	(1,5)				$(1, 10^{-4})$				$(10^{-4}$,$10^{-5})$			
n	10	20	40	80	10	20	40	80	10	20	40	80
Nbre d'itérations DN	1	1	1	1	12	12	12	12	14	14	14	14
Nbre d'itérations NN	1	1	1	1	22	22	22	22	23	23	23	23

Table 1: Results for the Dirichlet-Neumann algorithm and for the Neumann-Neumann algorithm.

Remark 8.2. *In the case of two domains Ω_1 and Ω_2 of the same size, we see that the number of iterations increases with the jump size of the coefficients.*

Even though the Dirichlet-Neumann(DN) algorithm converges in a smaller number of iterations than the Neumann-Neumann(NN) algorithm, the error calculation $|u_{SDDM} - u_{ADDM}\|_{mathbbL^2}$ with the stopping criterion (34), shows that the Neumann-Neumann algorithm gives a good approximation than Dirichlet-Neumann. The table (**??**) presents the results obtained for different meshes.

Taille du maillage (n)	10	20	40	80
ERREUR-DN	0,0439327	0,0251683	0,0130351	0,00657722
ERREUR-NN	0,000623169	0,000234582	8,43998E-005	2,99788E-005

Table 2: Error calculation for different meshes.

It is clear that the error of both algorithms improves while refining the mesh.

9 Conclusion

Domain decomposition methods are numerical methods well adapted to parallel solving of partial differential equations. They are not an alternative but an essential choice for solving large, coupled or geometrically complex problems. These methods are more efficient than an iterative method applied to the global problem, because they combine direct methods and an iterative method to solve an interface problem, better conditioned than the global problem.

The study made on this work has treated the domain decomposition methods without overlap. The mathematical foundations of domain decomposition methods applied to the Poisson equation were studied, then the study was extended to a heterogeneous problem. The two algorithms Dirichlet-Neumann and Neumann-Neumann were presented to solve these systems, and to illustrate the importance of these methods as preconditioning methods. The discretization of the problem has been done by finite elements, and the numerical implementation has been done on the FreeFem software.

Numerical results concerning the convergence speed and the error estimation are presented in the numerical part.

References

1. A. TOSELLI, O.WINDLUND. *Domain Decomposition Methods-Algorithms and Theory*.Springer Series in Computational Mathematics. Springer, Berlin, 2005.
2. A. QUARTERONI AND A. VALLI. *Domain Decomposition Methods for Partial Differential Equations*. Scientific Computation. Oxford University Press, Oxford, 1999.
3. D. VASSILEV, C. WANGB AND I. YOTOV, *Domain decomposition for coupled Stokes and Darcy flows*.Comput. Methods Appl. Mech. Engrg. 268 (2014) 264?283.
4. W.J LAYTON, F. SCHIEWECK AND I. YOTOV, *Coupling Fluid Flow With Pourous Media*. SIAM J. NUMER. ANAL. Vol. 40, No. 6, pp. 2195-2218.
5. G.ALLAIRE, *Analyse numérique et optimisation*. Éditions de l'École Polytechnique 2005.http://www.cmap.polytechnique.fr/ allaire/livre2.html.
6. V.CHABANNES, *Vers la simulation des écoulements sanguins*. https://tel.archives-ouvertes.fr/tel-00923731v2.
7. D.JEROME ; P.MARC,*Ecoulement à travers une couche poreuse limitée par une paroi perforée*. Institut national polytechnique de Toulouse, Toulouse, FRANCE 2000.
8. A.ERN,*Aide Mémoire Éléments fini*. Dunod, Paris, 2005.
9. G. BEAVERS AND D. JOSEPH ,*Boundary conditions at a naturally imperméable wall*. 30 Mars 1967 .
10. P. SAFFMAN,*On the boundary condition at the surface of a porous media*.Stud. Appl. Math., L. (1971) 93-101.

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